Effect of Modulation on Rayleigh-Benard Convection-II

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The linear stability of a horizontal fluid layer, confined between two rigid walls, heated from below and cooled from above is considered. The temperature gradient between the walls consists of a steady part and a periodic part that oscillates with time. Only infinitesimal disturbances are considered. Numerical results for the critical Rayleigh number are obtained for various Prandtl numbers and for various values of the frequency. Some comparisons with known results have also been made.

Key words: Modulation; Stability; Rayleigh Number; Odd Solution; Thermal Convection.

1. Introduction

This paper concerns the stability of a fluid layer confined between two horizontal planes and heated periodically from below and cooled from above. Chandrasekhar [1] and Drazin and Reid [2] have given comprehensive accounts of the various investigations of hydrodynamics and hydrodynamic stability. Koschmieder [3] and Getling [4] have also discussed in detail the Rayleigh-Benard convection.

Since the problems of Taylor stability and Benard stability are very similar, Venezian [5] investigated the thermal analogue of Donnelly's experiment [6], using free-free surfaces, and compared his results with the results of Donnelly. Venezian does not find any such finite frequency as obtained by Donnelly, but he finds that, for modulation only at the lower surface, the modulation would be stabilizing with maximum stabilization occurring as the frequency goes to zero. However it was suggested by Venezian that the linear stability theory ceases to be applicable when the frequency of modulation is sufficiently small.

Rosenblat and Herbert [7] have investigated the linear stability problem for free-free surfaces, using low-frequency modulation, and found an asymptotic solution. Periodicity and amplitude criteria were employed to calculate the critical Rayleigh number. Rosenblat and Tanaka [8] have used Galerkin's procedure to solve the linear problem for more realistic boundary conditions, i.e. rigid walls. A similar problem has been considered earlier by Gershuni and Zhukhovitskii [9] for a temperature profile obeying a rectangular law. Yih and Li [10] have investigated the formation of convective cells in a fluid between two horizontal rigid boundaries with time-periodic temperature distribution, using Floquet theory. They found that the disturbances (or convection cells) oscillate either synchronously or with half frequency.

Gresho and Sani [11] have treated the linear stability problem with rigid boundaries and found that gravitational modulation can significantly affect the stability limits of the system. Finucane and Kelly [12] have carried out an analytical-experimental investigation to confirm the results of Rosenblat and Herbert. Besides investigating the linear stability, Roppo et al. [13] have also carried out a weakly non-linear analysis of the problem. Aniss et al. [14] have worked out a linear problem of the convection parametric instability in the case of a Newtonian fluid confined in a Hele-Shaw cell and subjected to a vertical periodic motion. In their asymptotic analysis they have investigated the influence of the gravitational modulation on the instability threshold. Recently Bhatia and Bhadauria [15, 16], Bhadauria and Bhatia [17] have studied the linear stability problem for more general temperature profiles and compared their results with other results. More recently Bhadauria et al. [18] have studied the thermal convection in a Hele-Shaw cell with parametric excitation under saw-tooth and step-function oscillations.

The object of the present study is to find the critical conditions under which thermal convection starts. Here the temperature modulation between the plates is out of phase. Only odd solutions have been considered.

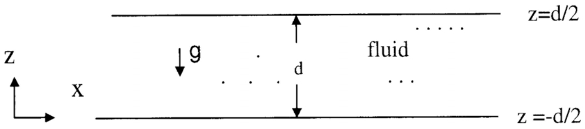

Fig. 1. Benard Configuration.

The results are relevant for convective flows in the terrestrial atmosphere.

2. Formulation

Consider a fluid layer of a viscous, incompressible fluid, confined between two parallel horizontal walls, one at $z = -d/2$ and the other at $z = d/2$. The walls are infinitely extended and rigid. The configuration is shown in the Figure 1.

The governing equations in the Boussinesq approximation are

$$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V} \cdot \nabla \boldsymbol{V} = -\frac{1}{\rho_\mathrm{m}} \nabla p + [1 - \alpha(T - T_\mathrm{m})] X + \nu \nabla^2 \boldsymbol{V}, \tag{2.1}$$

$$\nabla \cdot \mathrm{V} = 0, \tag{2.2}$$

$$\frac{\partial \mathrm{T}}{\partial \mathrm{t}} + \boldsymbol{V} \cdot \nabla T = \kappa \nabla^2 T, \tag{2.3}$$

where ρ_m, T_m are the constant reference density and temperature, respectively, $X = (0, 0, -g)$ is the acceleration due to gravity, ν the kinematic viscosity, κ the thermal diffusivity and α the coefficient of volume expansion, $\mathrm{V} = (u, v, w)$ is the fluid velocity. The relation between ρ_m and T_m is given by

$$\rho = \rho_\mathrm{m} [1 - \alpha(T - T_\mathrm{m})]. \tag{2.4}$$

To modulate the wall temperatures, the boundary conditions are

$$T(t) = \begin{cases} \beta d(1 + \epsilon \cos \omega t) \text{ at } z = -d/2, \\ \beta d \epsilon \cos \omega t \text{ at } z = d/2. \end{cases} \tag{2.5}$$

Here ω is the modulating frequency, $2\pi/\omega$ is the period of oscillation, ϵ represents the amplitude of modulation, β is thermal gradient. The equations (2.1 – 2.4) and (2.5) admit an equilibrium solution in which

$$\boldsymbol{V} = (u, v, w) = 0, \ T = \bar{T}(z, t), \ p = \bar{p}(z, t). \tag{2.6}$$

The equation for the pressure $\bar{p}(z, t)$, which balances the buoyancy force, is not required explicitly, however the temperature $\bar{T}(z, t)$ can be given by the diffusion equation

$$\frac{\partial \bar{T}}{\partial t} = \kappa \frac{\partial^2 \bar{T}}{\partial z^2}. \tag{2.7}$$

The differential equation (2.7) can be solved with the help of the boundary conditions (2.5). Consider

$$T(z, t) = T_\mathrm{S}(z) + \epsilon T_1(z, t) \tag{2.8}$$

where $T_\mathrm{S}(z)$ is the steady temperature field and $T_1(z, t)$ is the oscillating part.

Then the solution is

$$T_S(z) = \Delta T \left(\frac{1}{2} - \frac{z}{d} \right) \tag{2.9}$$

and

$$T_1(z, t) = -\Delta T \mathrm{Re} \left\{ \frac{F(z, t)}{\sinh(\lambda/2)} \right\}, \tag{2.10}$$

where

$$F(z, t) = \sinh(\lambda z/d) e^{i\omega t} \tag{2.11}$$

and

$$\lambda^2 = i\omega d^2/\kappa. \tag{2.12}$$

Here the object is to examine the behaviour of infinitesimal disturbances to the basic solution (2.6). With this in view substitute

$$\boldsymbol{V} = (u, v, w), \ T = \bar{T}(z, t) + \theta, \ p = \bar{p}(z, t) + p_1 \tag{2.13}$$

into (2.1) – (2.3). Scaling the length, time, temperature, velocity and pressure according to

$$z = d\, z', \ t = t'/\omega, \ \bar{T} = \beta\, d\, T_0, \ \theta = \beta\, d\, \theta',$$
$$\boldsymbol{V} = (\alpha\, g\, \beta\, d^3\, a^2/\nu) \boldsymbol{V}', \tag{2.14}$$
$$p_1 = (\alpha\, g\, \beta\, \kappa\, d^2\, \rho_\mathrm{m}/\nu) p',$$

the non-dimensionalized governing equations in linear form are

$$a^2\, \omega^* \frac{\partial \boldsymbol{V}}{\partial t} + \nabla p = P\, \theta\, \hat{\mathbf{k}} + a^2 P \nabla^2 \boldsymbol{V}, \tag{2.15}$$

$$\nabla \cdot \boldsymbol{V} = 0, \tag{2.16}$$

$$\omega^* \frac{\partial\theta}{\partial t} + R\,a^2 \left(\frac{\partial T_0}{\partial z}\right) w = \nabla^2\theta, \qquad (2.17)$$

where $P = \nu/\kappa$ is the Prandtl number, $R = \alpha g \Delta T d^3/\nu\kappa$ is the Rayleigh number, and $\hat{\mathbf{k}}$ is the vertical unit vector. $D = \frac{\partial}{\partial z}, F'(z,t) = DF(z,t)$, and $\omega^* = \omega d^2/\kappa$ is the non-dimensional frequency, which is a measure of the thickness of the thermal boundary layer at the planes. Here $a = \left(a_x^2 + a_y^2\right)^{1/2}$ is the horizontal wave number. In the above equations the primes have been omitted.

The temperature gradient $\partial T_0/\partial z$, obtained from the dimensionless form of (2.8) is

$$\frac{\partial T_0}{\partial z} = -1 - \epsilon \mathrm{Re}\left[\frac{F'(z,t)}{\sinh(\lambda/2)}\right], \qquad (2.18)$$

where

$$F'(z,t) = \lambda\cos(\lambda z)e^{it} \qquad (2.19)$$

and

$$\lambda^2 = i\omega^*. \qquad (2.20)$$

Henceforth the asterisk will be dropped and ω will be considered as the non-dimensional frequency. For convenience, the entire problem has been expressed in terms of w and θ. These quantities are Fourier analyzed with respect to their variations in the xy-plane. Write

$$w = w(z,t)\exp[i(a_x x + a_y y)], \qquad (2.21)$$

$$\theta = \theta(z,t)\exp[i(a_x x + a_y y)]. \qquad (2.22)$$

Now taking the curl of (2.15) twice and using (2.21) and (2.22), the system of equations reduces to

$$\omega\Big(\frac{\partial^2}{\partial z^2} - a^2\Big)\frac{\partial w}{\partial t} = -\mathrm{P}\theta + \mathrm{P}\Big(\frac{\partial^2}{\partial z^2} - a^2\Big)^2 w, \qquad (2.23)$$

$$\omega\frac{\partial\theta}{\partial t} = \Big(\frac{\partial^2}{\partial z^2} - a^2\Big)\theta - \mathrm{R}a^2\Big(\frac{\partial T_0}{\partial z}\Big)w. \qquad (2.24)$$

The boundary conditions on w and θ are

$$w = Dw = 0 \text{ at } \quad z = \pm\frac{1}{2}, \qquad (2.25)$$

$$\theta = 0 \text{ at } \quad z = \pm\frac{1}{2}. \qquad (2.26)$$

3. Method

From the expression (2.11) it is clear that $F(z,t)$ is an odd function of z so $F'(z,t)$ is an even function of z. By carefully analyzing (2.23) and (2.24) and the boundary conditions (2.25) and (2.26) one can see that the proper solution of (2.23) and (2.24) can be divided into two non-combining groups of even and odd solutions. Previous investigations on thermal convection have shown that disturbances corresponding to even solutions are most unstable; however here we discuss the stability of the disturbances corresponding to odd solutions.

Now, since θ vanishes at $z = \pm\frac{1}{2}$, it is expanded in a series of $\sin(2n\pi z)$. Also w is written in a series of ϕ_n so that

$$(D^2 - a^2)^2\phi_n = \sin(2n\pi z), \qquad (3.1)$$

where

$$\phi_n = D\phi_n = 0 \quad \text{at} \quad z = \pm\frac{1}{2}. \qquad (3.2)$$

Then the general solution of (3.1) can be given by [1, p. 58]

$$\phi_n = P_n \sinh az + Q_n z\cosh az + \gamma_n^2 \sin(2n\pi z), \qquad (3.3)$$

where

$$P_n = (-1)^n \frac{2n\pi\gamma_n^2}{\sinh a - a}\cosh(a/2), \qquad (3.4)$$

$$Q_n = -(-1)^n \frac{4n\pi\gamma_n^2}{\sinh a - a}\sinh(a/2) \qquad (3.5)$$

and

$$\gamma_n = \frac{1}{4n^2\pi^2 + a^2}. \qquad (3.6)$$

The expansions for w and θ can be written as

$$w(z,t) = \sum_{n=1}^{\infty} A_n(t)\phi_n(z), \qquad (3.7)$$

$$\theta(z,t) = \sum_{n=1}^{\infty} B_n(t)\sin(2n\pi z). \qquad (3.8)$$

Now substitute (3.7) and (3.8) into the equations (2.23) and (2.24), and multiply by $\sin(2m\pi z)$. The resulting equations are then integrated with respect to z in the interval $(-1/2,\ 1/2)$. The outcome is a system of ordinary differential equations for the unknown coefficients $A_n(t)$ and $B_n(t)$:

$$\omega \sum_{n=1}^{\infty} \left[K_{n\,m} - a^2 P_{n\,m}\right] \frac{\mathrm{d}A_n}{\mathrm{d}t} = -\frac{P}{2} B_m + P \sum_{n=1}^{\infty} \left[L_{n\,m} - 2a^2 K_{n\,m} + a^4 P_{n\,m}\right] A_n, \tag{3.9}$$

$$\frac{\omega}{2}\frac{\mathrm{d}B_m}{\mathrm{d}t} = -\frac{1}{2}\left[(2m\pi)^2 + a^2\right] B_m + \mathrm{Ra}^2 \sum_{n=1}^{\infty} \left[P_{n\,m} + \epsilon \mathrm{Re}\left\{G_{n\,m} e^{it}\right\}\right] A_n, \tag{3.10}$$

($m = 1, 2, 3, \ldots$).

The other coefficients, which occur in (3.9) and (3.10) are

$$P_{n\,m} = \int_{-1/2}^{1/2} \phi_n(z) \sin(2m\pi z)\mathrm{d}z, \tag{3.11}$$

$$K_{n\,m} = \int_{-1/2}^{1/2} D^2\phi_n(z) \sin(2m\pi z)\mathrm{d}z, \tag{3.12}$$

$$L_{n\,m} = \int_{-1/2}^{1/2} D^4\phi_n(z) \sin(2m\pi z)\mathrm{d}z, \tag{3.13}$$

and

$$G_{n\,m} = \frac{\lambda}{\sinh(\lambda/2)} \cdot \int_{-1/2}^{1/2} \phi_n(z) \cosh(\lambda z) \sin(2m\pi z)\mathrm{d}z. \tag{3.14}$$

Here the integrals (3.11) – (3.13) have been obtained in their closed forms, however (3.14) has been calculated numerically, using Simpson's (1/3)rd rule [19, p. 125]. Thus

$$P_{n\,m} = \frac{1}{2}\gamma_n^2 \delta_{n\,m} + (-1)^m 2m\pi\gamma_m \left[2P_n \sinh(a/2) + Q_n\left\{\cosh(a/2) - 4a\gamma_m \sinh(a/2)\right\}\right], \tag{3.15}$$

$$K_{n\,m} = -\frac{1}{2}\gamma_n^2 (2n\pi)^2 \delta_{n\,m} + (-1)^m 2m\pi\gamma_m \left[2(a^2 P_n + 2aQ_n)\sinh(a/2) + a^2 Q_n\left\{\cosh(a/2) + 4a\gamma_m \sinh(a/2)\right\}\right], \tag{3.16}$$

$$L_{n\,m} = \frac{1}{2}\gamma_n^2 (2n\pi)^4 \delta_{n\,m} + (-1)^m 2m\pi\gamma_m \left[2a^4 P_n \sinh(a/2) + Q_n\left\{4a^3(2 - a^2\gamma_m)\sinh(a/2) + a^4\cosh(a/2)\right\}\right], \tag{3.17}$$

where $\delta_{n\,m}$ is the Kronecker delta.

It is convenient for computational purpose to take $m = 1, 2, 3, \ldots, N$ i.e. total $2N$ equations and then rearrange them. For this, first multiply (3.9) by the inverse of the matrix $(K_{n\,m} - a^2 P_{n\,m})$, and then introduce the notations

$$x_1 = A_1,\ x_2 = B_1,\ x_3 = A_2,\ x_4 = B_2, \cdots. \tag{3.18}$$

Now combine (3.9) and (3.10) to the form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = H_{i1}x_1 + H_{i2}x_2 + \cdots + H_{iL}x_L, \tag{3.19}$$

($i = 1, 2, 3, \ldots, 2N$ and $L = 2N$),

where $H_{ij}(t)$ is the matrix of the coefficients in (3.9) and (3.10). Since the coefficients $H_{ij}(t)$ are periodic in t with period 2π, we can discuss the stability of the solution of (3.19) on the basis of the classical Floquet theory [21, p. 55]. Let

$$x_n(t) = x_{in}(t) = \mathrm{col}\left[x_{1n}(t), x_{2n}(t), \cdots, x_{Ln}(t)\right],$$

$$(n = 1, 2, 3, \ldots, 2N) \tag{3.20}$$

be the solutions of (3.19) which satisfy the initial conditions

$$x_{in}(0) = \delta_{in} \tag{3.21}$$

The solutions (3.20) with the conditions (3.21) form $2N$ linearly independent solutions of (3.19). Once these solutions are found, one can get the values of $x_{in}(2\pi)$ and then arrange them in the constant matrix

$$C = [x_{in}(2\pi)]\,. \tag{3.22}$$

The eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_L$ of the matrix C are also called the characteristic multipliers of the system (3.19), and the numbers μ_r, defined by the relations

$$\lambda_r = \exp(2\pi\mu_r), \quad r = 1, 2, 3, \ldots, 2N \tag{3.23}$$

are the characteristic exponents.

The values of the characteristic exponents determine the stability of the system. We assume that the μ_r are ordered, so that

$$\mathrm{Re}(\mu_1) \geq \mathrm{Re}(\mu_2) \geq \cdots \geq \mathrm{Re}(\mu_L). \tag{3.24}$$

Then the system is stable if $\mathrm{Re}(\mu_1) < 0$, while $\mathrm{Re}(\mu_1) = 0$ corresponds to one periodic solution and represents a stability boundary. This periodic disturbance is the only disturbance which will manifest itself at marginally stability.

To obtain the matrix C we have integrated the system (3.19) using the Runge-Kutta-Gill Procedure [19, p. 217]. The eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_L$ of the matrix C are found with the Rutishauser method [21, p. 116].

4. Results and Discussion

The first approximation to the critical Rayleigh number in the absence of modulation ($\epsilon = 0$) is found by setting $n = 1$, $m = 1$ in (3.9) and (3.10). This corresponds to $sin(2\pi z)$, a trial function for θ. The corresponding value for R is

$$R = (4\pi^2 + a^2)^3 \Big/ a^2 \big[1 - 64a\pi^2 \sinh^2(a/2) \tag{4.1}$$

$$/\big\{(4\pi^2 + a^2)^2(\sinh a - a)\big\}\big].$$

This gives

$$R = 17803.24 \quad \text{at} \quad a = 5.365, \tag{4.2}$$

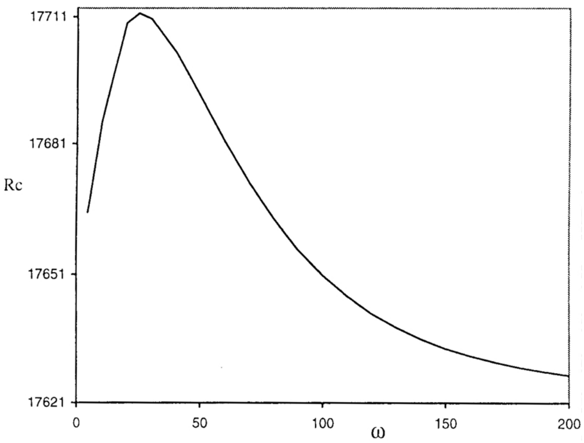

Fig. 2. Variation of R_c with ω. $P = 0.1$, $\epsilon = 0.1$, $a = 3.1168$.

which is in contrast to the exact value of 17610.39 at the same wavenumber. By including more terms in the expansion of w and θ one can achieve a higher degree of accuracy. The second approximation to the Rayleigh number is found to be 17621.74 at $a = 5.363808$, which is obtained by setting $m, n = 1$ and 2. These values equal, as they should, Chandrasekhar's [1] values.

The modified value of R_c with variation of other parameters have been calculated, when $\epsilon \neq 0$. The critical wavenumber a has also been checked. Here the results have been obtained by solving (3.19) for x_1, x_2, x_3, and x_4. The results are calculated for moderate values of ϵ. As we are interested only in the modulating effect of the oscillation, there seems to be no reason why this theory can not be applied for large values of the parameters.

Figure 2 depicts the value of R_c as a function of the modulating frequency ω. One can see that the effect of modulation of wall temperatures is stabilization. R_c increases as ω increases, reaches its maximum at some modulating frequency ω and then decreases and tends to the steady value as ω tends to infinity. This result agrees with the result of Donnelly [6], who found while modulating the angular speed of the inner cylinder that the degree of stabilization rises from zero at high frequency to a maximum at a frequency of $0.274(\nu/d^2)$, where d is the gap between the cylinders and ν is the kinematic viscosity. Here this frequency is found to be at $\omega = 25$.

Also, from Figs. 3 and 4 one can see that the effect of the unsteady part of the primary temperature is stabilizing. The stabilization is maximum near $\omega = 0$

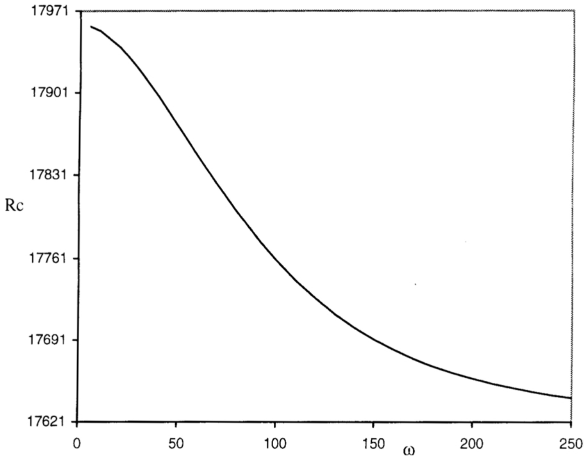

Fig. 3. Variation of R_c with ω. $P = 0.73$, $\epsilon = 0.1$, $a = 3.1168$.

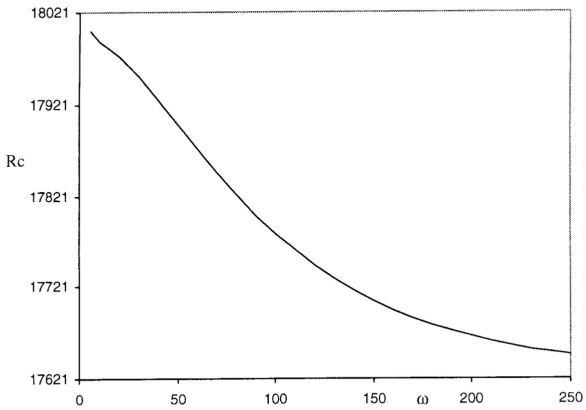

Fig. 4. Variation of R_c with ω. $P = 1.0$, $\epsilon = 0.1$, $a = 3.1168$.

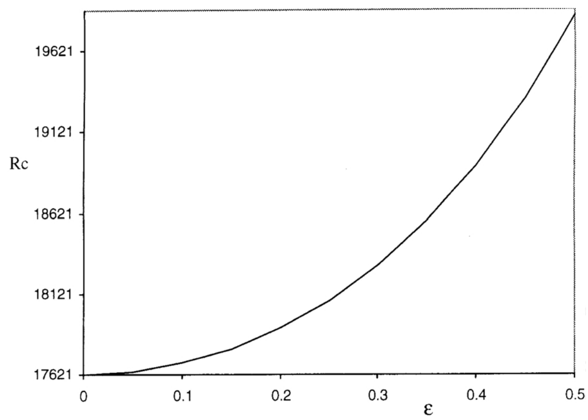

Fig. 5. Variation of R_c with ϵ. $P = 0.1$, $\omega = 50.0$, $a = 3.1168$.

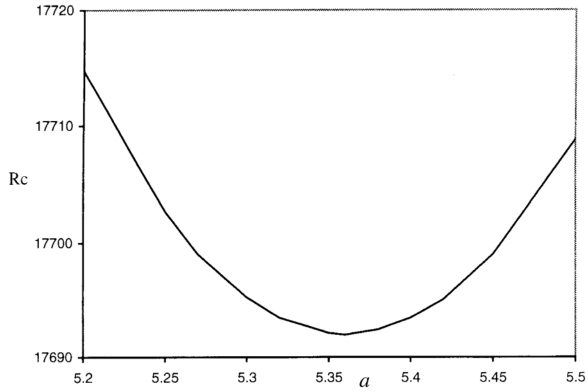

Fig. 6. Variation of R_c with a. $P = 0.1$, $\omega = 50.0$, $\epsilon = 0.1$.

and disappears when the ω is sufficiently large. This agrees with the results of Venezian [5], Rosenblat and Tanaka [8], Bhatia and Bhadauria [15], and Bhadauria and Bhatia [16]. This also agrees with the results of Yih and Li [10] who found, while studying the instability of unsteady flows, that the effect of modulation is stabilizing. The results also agree with Donnelly's [6] findings for the related problem of Taylor vortices, that oscillation of one cylinder can only stabilize the couette flow.

When the modulating frequency is small, the convective wave propagates across the fluid layer, thereby inhibiting the instability, and so the convection occurs at higher Rayleigh number than that predicted by the linear theory for a steady temperature gradient.

Figure 5 shows the variation of R_c with the amplitude of modulation. It is found that, as the amplitude of modulation increases, R_c also increases, showing the stabilizing effect.

Finally, in Fig. 6, the variation of R_c with the wavenumber a has been depicted. It is very clear from the figure that the critical value of the wavenumber a is found to be near 5.36.

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